

HODGE-THEORETIC OBSTRUCTION TO EXISTENCE OF QUATERNION ALGEBRAS

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1. INTRODUCTION

The subject of this paper is the Brauer group of a nonsingular complex projective variety. More specifically, we study the question of whether a 2-torsion element of the cohomological Brauer group is representable by a quaternion algebra over the generic point. Using intersection theory – on schemes and on algebraic stacks – we are able to describe an obstruction to such a representation, and therefore, to give examples of varieties with 2-torsion classes that are not representable by quaternion algebras.

Let X be a nonsingular complex algebraic variety. It is a well-known consequence of the Grauert–Remmert theorem, plus GAGA, that topological 2-sheeted covers, classified by $H^1(X, \mathbf{Z}/2)$, correspond exactly to algebraic degree 2 unramified covers. There is similarly an algebraic object that we can associate to an element $\alpha \in H^2(X, \mathbf{Z}/2)$ – a kind of algebraic stack over X known as a *gerbe*. The element α determines a 2-torsion element of $\mathrm{Br}(X) = H^2(X, \mathbf{G}_m)$, and if this element is represented by a quaternion algebra then the quaternion algebra can be identified with the (descent to X of the) endomorphism algebra of a rank 2 vector bundle on the gerbe. The second Chern class of this bundle (or rather, a specific multiple of this) is an algebraic class on X , and if X is projective this leads to a Hodge-theoretic obstruction to representing the element of $\mathrm{Br}(X)$ by a quaternion algebra.

We turn now to the statement of our main result.

Theorem 1. *Let X be a nonsingular complex projective variety. Let $\alpha \in H^2(X, \mathbf{Z}/2)$ be a class such that (i) the image of α in the Brauer group of the function field $\mathrm{Br}(k(X))$ is representable by a quaternion algebra over $k(X)$, and (ii) there exists a preimage $\alpha_0 \in H^2(X)$ under the natural map $H^2(X) = H^2(X, \mathbf{Z}) \rightarrow H^2(X, \mathbf{Z}/2)$. Then there exists an algebraic class $\eta \in A^2 X$ whose cycle class in cohomology satisfies*

$$\mathrm{cl}(\eta) = 4(\alpha_0^2 + 2\varepsilon)$$

for some $\varepsilon \in H^4(X)$.

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An immediate consequence of Theorem 1 is the following result.

Corollary 1. *Let X be a nonsingular complex projective variety. Let N denote the Neron–Severi group of X , and $i: (H^2(X)/N) \otimes \mathbf{Z}/2 \rightarrow \mathrm{Br}(k(X))$ the injective homomorphism to the Brauer group induced by reduction of coefficients $H^2(X) \rightarrow H^2(X, \mathbf{Z}/2)$, followed by the natural map to the Brauer group. Let $M = H^4(X) \cap H^{2,2}(X)$; then for any $\beta \in (H^2(X)/N) \otimes \mathbf{Z}/2$, the image of β under the homomorphism*

$$(1) \quad (H^2(X)/N) \otimes \mathbf{Z}/2 \rightarrow (H^4(X)/M) \otimes \mathbf{Z}/2$$

given by $\beta \mapsto \beta^2$ is an obstruction to representing $i(\beta)$ by a quaternion algebra over $k(X)$.

After stating conventions and preliminary results in Section 2, we prove Theorem 1 in Section 3. In Section 4, we provide examples for which the obstruction map (1) is nontrivial. The geometric significance of a nontrivially obstructed class β is that $i(\beta)$ cannot be represented by a Brauer–Severi variety of relative dimension 1, even over the generic point of X .

A deep result of Merkurjev [8] asserts that if F is any field with $\mathrm{char} F \neq 2$, then the 2-torsion subgroup of the Brauer group is generated by the classes of quaternion algebras. Then we say F is *linked* if every 2-torsion element of $\mathrm{Br}(F)$ is representable by a quaternion algebra. Some classes of fields known to be linked are listed in [4]. This list includes all fields of transcendence degree 2 over \mathbf{C} ; it is shown in [12] that more generally any C_2 -field is linked. While [4] gives some explicit examples of non-linked fields, the important main result in that paper there is a necessary (but not sufficient) characterization of linked fields by their u -invariant, and this provides a way to generate unlimited numbers of examples of non-linked fields.

In Section 4 we present a result, Theorem 2, which shows that function fields of complex algebraic threefolds are not, in general, linked. The threefolds we study are Brauer–Severi varieties over surfaces, and for such X there are classes in $H^2(X, \mathbf{Z}/2)$ for which the induced elements of $\mathrm{Br}(k(X))$ are not representable by quaternion algebras over $k(X)$. So, our examples differ from previously known examples in that the constructions are of a geometric nature.

As a consequence of cohomological purity and the valuative criterion for properness, the image of the inclusion $\mathrm{Br}(X) \rightarrow \mathrm{Br}(k(X))$ can be identified with the unramified Brauer group [11] of $k(X)$. Thus, the cohomological Brauer group is a birational invariant, for nonsingular complete varieties over a given field of characteristic zero. It is therefore a weakness of the present approach to require a specific nonsingular projective model to study a problem which is intrinsic to the function field.

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2. PRELIMINARIES

All schemes are of finite type over a field. All stacks are algebraic and of finite type over a field. A_* denotes the Chow groups of a stack as in [6]. For a smooth stack X of pure dimension n , we let $A^k X$ denote $A_{n-k} X$. For a topological space X , $H^*(X)$ denotes $H^*(X, \mathbf{Z})$. When X is a complex variety (respectively, a reduced algebraic stack of finite type over $\text{Spec } \mathbf{C}$), $H^*(X)$ denotes the cohomology of the underlying analytic space (respectively, a topological realization of the underlying analytic stack). Relevant facts on stacks can be found in [1, 7]. $\text{Br}(X)$ denotes the cohomological Brauer group of X ; facts on Brauer groups can be found in [5, 9].

Proposition 1. *Let X be a regular algebraic stack of pure dimension n , and assume X contains an open substack U such that (i) the complement $Z = X \setminus U$ is empty or of codimension ≥ 2 in X , and (ii) U is isomorphic to a quotient stack of the form $[U/G]$, where U is an algebraic space and G is a linear algebraic group. Then the first Chern class homomorphism $\text{Pic}(X) \rightarrow A_{n-1} X$ is an isomorphism.*

Proof. There are obvious isomorphisms $\text{Pic}(X) \rightarrow \text{Pic}(U)$ and $A_{n-1} X \rightarrow A_{n-1} U$, compatible with the first Chern class map, so it suffices to consider the case $X = U$ is a quotient stack, and we are reduced to [2, Thm. 1]. \square

Corollary 2. *Let X be a regular algebraic stack of pure dimension n with finite stabilizers at all geometric points. Then $\text{Pic}(X) \simeq A_{n-1} X$.*

Proof. A finite cover by a scheme $Y \rightarrow X$ exists by [3, Thm. 2.8] (or in the case of primary interest – X a Deligne-Mumford stack – by [7, Thm. 16.6]). We may suppose Y normal, hence finite flat over X away from a closed substack Z of X which is empty or of codimension ≥ 3 . This implies (cf. [6, Prop. 3.5.7]) that $U := X \setminus Z$ is a quotient stack, and Proposition 1 applies. \square

Results that follow use the fact that gerbes over a scheme X , banded by the group of n^{th} roots of unity μ_n , are classified by $H^2(X, \mu_n)$ [9, §IV.2]. When n is invertible in the base field (the case of primary interest to us), this classifying group is an étale cohomology group; in general, cohomology for the flat (fppf) topology must be employed.

Proposition 2. *Let X be a regular scheme, and let n be a positive integer. Let $\beta \in H^2(X, \mu_n)$, and let \mathcal{G} be the gerbe over X , banded by μ_n , classified by β . Then the following are equivalent:*

- (i) *The gerbe $\mathcal{G} \rightarrow X$ is Zariski locally trivial.*
- (ii) *\mathcal{G} is a trivial gerbe over some nonempty Zariski open subset of X .*
- (iii) *The image of β in $\text{Br}(k(X))$ is zero.*

(iv) *There exists a line bundle on \mathcal{G} on which the action of stabilizer groups at geometric points of \mathcal{G} is faithful.*

Proof. Clearly, (i) implies (ii), and (ii) implies (iii). As $\mathrm{Br}(X) \rightarrow \mathrm{Br}(k(X))$ is injective, (iii) implies β lies in the image of the boundary homomorphism

$$\delta: H^1(X, \mathbf{G}_m) \rightarrow H^2(X, \mu_n)$$

of the Kummer sequence. If $\beta = \delta(\alpha)$, then \mathcal{G} is isomorphic to a \mathbf{G}_m -quotient of the principal bundle on X associated to α . So, (iii) implies (iv). Given a line bundle as in (iv), we can identify \mathcal{G} with a \mathbf{G}_m -quotient of the associated principal bundle P . Now $P \rightarrow X$ is the principal bundle of a class $\alpha \in H^1(X, \mathbf{G}_m)$ with $\delta(k\alpha) = \beta$ for some k prime to n , and (i) holds. \square

Corollary 3. *Let p be a prime. Let X be a regular scheme, $\beta \in H^2(X, \mu_p)$, and let $f: \mathcal{G} \rightarrow X$ be the gerbe banded by μ_p with class β . Then*

$$f^*: \mathrm{Pic}(X) \rightarrow \mathrm{Pic}(\mathcal{G})$$

is an isomorphism if and only if the image of β in $\mathrm{Br}(k(X))$ is nonzero.

Proposition 3. *Let n be a positive integer. Let X be a scheme, and let \mathcal{G} be the gerbe over X , banded by μ_n , classified by $\beta \in H^2(X, \mu_n)$. Now suppose β is in the image under the boundary homomorphism of the Kummer sequence. If E is a line bundle on X whose class in $\mathrm{Pic}(X)$ maps under the boundary homomorphism to β , then for each k there is an exact sequence*

$$\bigoplus_{s \geq 1} A_{k+s} X \rightarrow \bigoplus_{s \geq 0} A_{k+s} X \rightarrow A_k \mathcal{G} \rightarrow 0$$

where the map on the left is

$$(\alpha_1, \alpha_2, \dots) \mapsto (c_1(E) \cap \alpha_1, c_1(E) \cap \alpha_2 - n\alpha_1, \dots).$$

Proof. We follow the program of Equivariant Intersection Theory [2]: \mathcal{G} is a \mathbf{G}_m -quotient of E minus the zero section, so scheme approximations to \mathcal{G} are complements of zero sections of line bundles over $X \times \mathbf{P}^r$:

$$A_k \mathcal{G} = A_{k+r+1}((E \otimes \mathcal{O}_{\mathbf{P}^r}(-n) \setminus s(X \times \mathbf{P}^r))$$

for r sufficiently large. The result now follows from the excision sequence

$$A_{k+r+1}(X \times \mathbf{P}^r) \rightarrow A_{k+r+1}(E \otimes \mathcal{O}_{\mathbf{P}^r}(-n)) \rightarrow A_k \mathcal{G} \rightarrow 0. \quad \square$$

Corollary 4. *If the conditions of Proposition 3 are met and additionally X is smooth, then there is a natural map*

$$A^*(X)[z]/(nz - c_1(E)) \rightarrow A^* \mathcal{G},$$

which is an isomorphism of rings.

An analogous result to Corollary 4 in the topological category is Proposition 4, which we give below after a topological lemma.

Lemma 1. *Let X be a paracompact topological space, and let E be a topological complex line bundle on X , with zero section s . Then there is a natural injective homomorphism for every k*

$$H^k(X)/(c_1(E) \cup H^{k-2}(X)) \rightarrow H^k(E \setminus s(X)),$$

and this is an isomorphism precisely when $c_1(E) \cup -: H^{k-1}(X) \rightarrow H^{k+1}(X)$ is injective.

Proof. We compactify E by setting $P = \mathbf{P}(E \oplus 1)$, the (complex) projectivization of the Whitney sum of E and a trivial complex line bundle. Now we have E and $F := P \setminus s(X)$, complex line bundles over X , with $E \cap F = E \setminus s(X)$. The Mayer-Vietoris sequence in cohomology gives

$$(2) \quad H^k(P) \rightarrow (H^k(X))^2 \rightarrow H^k(E \setminus s(X)) \rightarrow H^{k+1}(P) \rightarrow (H^{k+1}(X))^2.$$

Let L denote the tautological complex line bundle on P ; then the projective bundle theorem dictates $H^k(X) \oplus H^{k-2}(X) \simeq H^k(P)$ via $(\alpha, \beta) \mapsto \alpha + (c_1(L) \cup \beta)$. Since L is trivial on E and $L|_{P \setminus E} \simeq E$, the leftmost map in (2) is

$$(\alpha, \beta) \mapsto (\alpha, \alpha + (c_1(E) \cup \beta)).$$

The result is now immediate. \square

Fix now a realization of the classifying space $B(\mathbf{Z}/n)$ as a topological abelian group [10]. Then principal $B(\mathbf{Z}/n)$ -bundles are classified by $H^2(-, \mathbf{Z}/n)$.

Proposition 4. *Let X be a paracompact topological space, and assume X is homotopy equivalent to a CW complex. Suppose $\beta \in H^2(X, \mathbf{Z}/n)$ is the image of $\beta_0 \in H^2(X)$. Let $G \rightarrow X$ be a principal $B(\mathbf{Z}/n)$ -bundle classified by β . Then there is an injective ring homomorphism*

$$H^*(X)[u]/(nu - \beta_0) \rightarrow H^*(G),$$

which is an isomorphism in degrees ≤ 2 .

Proof. Let $\mathcal{O}_{\mathbf{CP}^\infty}(-n)$ denote the n^{th} twist of the tautological line bundle on \mathbf{CP}^∞ . The complement of the zero section in $\mathcal{O}_{\mathbf{CP}^\infty}(-n)$ is homotopy equivalent to $B(\mathbf{Z}/n)$, so by standard arguments, if L is a complex line bundle on X with $c_1(L) = \beta_0$, then the complement, over $X \times \mathbf{CP}^\infty$, of the zero section of $L \otimes \mathcal{O}_{\mathbf{CP}^\infty}(-n)$ is homotopy equivalent to G . Now the result follows from Lemma 1, plus the general fact that $H^0(X)$ and $H^1(X)$ are torsion free. \square

Proposition 5. *Let X be a scheme of pure dimension k , let n be a positive integer, and let $f: \mathcal{G} \rightarrow X$ be a gerbe banded by μ_n . Then for any $\alpha \in A_{k-1}\mathcal{G}$, we have $n\alpha = f^*\beta$ for some $\beta \in A_{k-1}X$.*

Proof. We may assume X is reduced. Consider the components of the regular locus $X^{\text{reg}} = \coprod X_i$. For each i , if we let f_i denote the restriction of f over X_i , then we claim the image of $f_i^*: A_{k-1}X_i \rightarrow A_{k-1}\mathcal{G}_i$ contains $nA_{k-1}\mathcal{G}_i$. Indeed, this follows by Proposition 3 when f_i is Zariski locally trivial; otherwise by Corollary 3 combined with Corollary 2, f_i induces an isomorphism of Chow groups in dimension $k-1$. Now the desired result follows by comparing the excision sequences for $X^{\text{reg}} \subset X$ and $\mathcal{G}^{\text{reg}} \subset \mathcal{G}$. \square

3. PROOF OF THE MAIN THEOREM

Let us start by recalling conditions (i) and (ii) on $\alpha \in H^2(X, \mathbf{Z}/2)$ (étale cohomology, which by cohomology comparison, equals singular cohomology). Condition (i), the existence of a quaternion algebra, is equivalent to existence of a Brauer–Severi variety of dimension 1 representing the class of α in $\text{Br}(k(X))$. This spreads out over a nonempty open subset of X and corresponds to a rank 2 vector bundle on an open subset of the $\mathbf{Z}/2$ -gerbe on X classified by α . Since X is regular, the vector bundle – and hence also the Brauer–Severi variety – can be extended over $X \setminus Z$ for some closed $Z \subset X$, empty or of codimension ≥ 3 .

A Brauer–Severi variety determines a class in the Brauer group, which is the obstruction to identification with the projectivization of an algebraic vector bundle. The analogous topological obstruction lies in the group $H^3(X)$. Requiring as in (ii) that α lifts to some $\alpha_0 \in H^2(X)$ is equivalent to requiring that the topological obstruction vanishes. So the setting of the theorem is the case where the obstruction vanishes topologically but not algebraically.

We fix some notation. We denote by \mathcal{G} the gerbe on X , banded by $\mathbf{Z}/2$, classified by α . We let V denote a Brauer–Severi variety over $X \setminus Z$, and E a rank 2 vector bundle on $\mathcal{G}_{X \setminus Z} := \mathcal{G} \times_X (X \setminus Z)$ such that $\mathbf{P}(E) \simeq V \times_X \mathcal{G}$. The restrictions of \mathcal{G} , V , and E over the generic point $\text{Spec } k(X)$ are denoted \mathcal{G}_{gen} , V_{gen} , and E_{gen} , respectively. We note that $\mathcal{G}_{X \setminus Z}$ is a quotient stack of a smooth variety by a linear algebraic group (e.g., of the principal bundle associated to E by the group GL_2), and hence $A^*\mathcal{G}_{X \setminus Z}$ and $H^*(\mathcal{G}_{X \setminus Z})$ can be identified with equivariant Chow groups and equivariant cohomology groups, respectively.

Now we proceed with the proof. The conic V_{gen} is not rational. So, the Chow ring of V_{gen} is $\mathbf{Z}[y]/(y^2)$, with y the class of a degree 2 point on V_{gen} . Comparing the projective bundle formula and Corollary 4, we have

$$A^*\mathcal{G}_{\text{gen}}[w]/(w^2 - c_1(E_{\text{gen}})w + c_2(E_{\text{gen}})) \simeq \mathbf{Z}[y, z]/(y^2, 2z - ky),$$

for some integer k . We have $A^1\mathcal{G}_{\text{gen}} = 0$ by Corollary 3. Hence $A^1\mathbf{P}(E_{\text{gen}}) \simeq \mathbf{Z}$, so k must be odd, and without loss of generality we can suppose $k = 1$. Then w and z both generate $A^1\mathbf{P}(E_{\text{gen}})$, so $w = \pm z$, and now

$$(3) \quad A^2\mathcal{G}_{\text{gen}} \simeq \mathbf{Z}/4,$$

with $c_2(E_{\text{gen}})$ as a generator.

Topologically, we have by Proposition 4 an injective ring homomorphism

$$H^*(X)[u]/(2u - \alpha_0) \rightarrow H^*(\mathcal{G}),$$

which is an isomorphism in degrees ≤ 2 . Moreover, the obstruction to writing V as the projectivization of a complex rank 2 vector bundle vanishes topologically, so we may write

$$V \simeq \mathbf{P}(B)$$

where B is a topological rank 2 complex vector bundle on $X \setminus Z$. Let f denote the restriction of the projection $\mathcal{G} \rightarrow X$, over $X \setminus Z$. We have $\mathbf{P}(f^*B) \simeq \mathbf{P}(E)$, and hence

$$f^*B \simeq E \otimes L$$

for some topological complex line bundle L on $\mathcal{G}_{X \setminus Z}$. Comparing the restrictions over a point of X , we see that L must be nontrivial on fibers of f . So,

$$(4) \quad c_1(L) = u + f^*\delta$$

for some $\delta \in H^2(X \setminus Z) \simeq H^2(X)$. The cycle class map to equivariant cohomology [2] respects Chern classes; using $A^1\mathcal{G} \simeq A^1X$ (Corollary 3), we see

$$c_2(E) = -c_1(L)^2 - c_1(E)c_1(L) + f^*c_2(B) = -u^2 + uf^*\beta + f^*\varepsilon$$

for some $\varepsilon \in H^4(X \setminus Z) \simeq H^4(X)$ and $\beta \in H^2(X)$. Hence

$$2c_2(E) + 2u^2 = f^*\beta'$$

in $H^4(\mathcal{G}_{X \setminus Z})$, for some $\beta' \in H^4(X)$.

Let $\gamma = c_2(E) \in A^2\mathcal{G}_{X \setminus Z} \simeq A^2\mathcal{G}$. By (3), $4c_2(E_{\text{gen}}) = 0$ in $A^2\mathcal{G}_{\text{gen}}$, and hence 4γ vanishes in $A^2\mathcal{G}_U$ for some nonempty open $U \subset X$. Let $Y = X \setminus U$; we may assume Y has pure dimension $n - 1$, where $n = \dim X$. Let i denote the inclusion $\mathcal{G}_Y \rightarrow \mathcal{G}$. By excision, 4γ lies in the image of $i_*: A_{n-2}\mathcal{G}_Y \rightarrow A_{n-2}\mathcal{G}$. By Proposition 5, now, we have

$$8\gamma = f^*\eta$$

for some $\eta \in A^2X$. It follows that

$$f^*\text{cl}(\eta) = -2f^*\alpha_0^2 + 4f^*\beta'.$$

The kernel of $f^*: H^4(X) \simeq H^4(X \setminus Z) \rightarrow H^4\mathcal{G}_{X \setminus Z}$ is 2-torsion, so

$$4\alpha_0^2 - 8\beta' \in \text{Im}(\text{cl}: A^2X \rightarrow H^4(X)),$$

and the theorem is proved.

4. EXAMPLE

We show that if X is any nonsingular complex projective surface with $p_g(X) \neq 0$, then there exist Brauer–Severi varieties $V \rightarrow X$ such that the obstruction map (1) of Corollary 1 for V is nontrivial.

Theorem 2. *Let X be a nonsingular complex projective surface with nonzero geometric genus. Let β be an element of $H^2(X)$ whose reduction to $H^2(X, \mathbf{Z}/2)$ maps to some nonzero $\lambda \in \text{Br}(X)$. If $V \rightarrow X$ is a smooth conic representing λ (this exists since X is a nonsingular surface over \mathbf{C}), then the obstruction map of Corollary 1 for the variety V is nontrivial. In fact, any element of $H^2(V)$ which generates $H^2(V)/H^2(X) \simeq \mathbf{Z}$ has nonzero image under the obstruction map.*

Proof. Denote by r the Picard number of X ; then r is strictly less than the second Betti number k of X . Let N be the Neron-Severi group of X . Without loss of generality, we may suppose the image of β in $H^2(X)/N$ is a primitive lattice element. Then we can write

$$H^2(X) = N \oplus \langle \beta \rangle \oplus T,$$

where T has rank $k - r - 1$.

Recall that the hypotheses dictate that V is topologically (but not algebraically) the projectivization of a rank 2 complex vector bundle on X . To compute the obstruction map for V , we use the constructions and notations of the proof of Theorem 1 applied to X and β . In particular, B denotes a topological complex rank 2 vector bundle such that $V \simeq \mathbf{P}(B)$, and hence

$$H^*(V) = H^*(X)[x]/(x^2 - c_1(B)x + c_2(B)).$$

Consider the following claim: the Picard number of V is $r + 1$ and the \mathbf{Q} -span of the Neron-Severi group of V is the \mathbf{Q} -span of N together with the element $2x + \gamma$, for some $\gamma \in H^2(X)$. Given this, then, in

$$H^4(V) = H^4(X) \oplus xN \oplus \langle x\beta \rangle \oplus xT,$$

we have

$$H^4(V) \cap H^{2,2}(V) = H^4(X) \oplus xN,$$

since for any $\delta \in N$, $2(x\delta) = (2x + \gamma)\delta - \gamma\delta \in H^{2,2}(V)$.

Recall that we have $f^*B \simeq E \otimes L$. Now $x^2 = xc_1(B) - c_2(B) = xc_1(E) + 2xc_1(L) - c_2(B)$, and by (4), $2c_1(L) \in \beta + 2H^2(X)$, so the image of x under the obstruction map (1) is nonzero. The obstruction map for V is thus completely determined by its vanishing on elements of $H^2(X)$ and its nonvanishing on x .

It remains to verify the claim. $\text{Pic}(\mathcal{G} \times_X V)$ is an extension of $\text{Pic}(\mathcal{G})$ by the free group generated by $c_1(\mathcal{O}_{\mathbf{P}(E)}(1))$. So $\text{Pic}(V)$ is an extension of $\text{Pic}(X)$ by some class which pulls back to $2c_1(\mathcal{O}_{\mathbf{P}(E)}(1))$ on $\mathcal{G} \times_X V$. But $2c_1(\mathcal{O}_{\mathbf{P}(E)}(1))$ differs from $2c_1(\mathcal{O}_{\mathbf{P}(f^*B)}(1))$ (the pullback of $2x$) by $2c_1(L)$, which lies in $H^2(X)$.

So, indeed, the Picard number of V is $r + 1$, and for some $\gamma \in H^2(X)$, $2x + \gamma$ is algebraic on V . \square

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